Algebra problem solving seminar

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1. A square matrix over a field of zero characteristic is nilpotent if and only if the traces of all its powers are 0.

2. Let H be a subgroup of the finite group G such that its index is the minimal prime divisor of the size of G. Then H is normal in G.

3. If $1 - ab$ is invertible in a ring, then $1 - ba$ is also invertible.

4. Let A and B be groups such that A is a homomorphic image of B and B is a homomorphic image of A. Are they necessarily isomorphic?

5. Can you obtain every Abelian group as the additive group of a ring with identity?

6. Prove that every element of a finite field can be written as the sum of two squares.

7. Let G be a 2-transitive permutation group on n points and let H be a subgroup of G of index less than n. Show that H is transitive.

8. Let G be a group and N a normal subgroup in G. Show that if both N and G/N are locally finite, then G is locally finite as well. (A group is locally finite if all its finitely generated subgroups are finite.)

9. Let A, B be square matrices over an algebraically closed field of characteristic zero, such that all linear combinations of A and B are diagonalizable. Show that A and B commute.

10. Let Γ be a finitely generated group. Then every generating set for Γ contains a finite generating set.

11. Let $R \leq \mathbb{C}$ be a finitely generated subring. Then the intersection of the finite index ideals of R is 0.

12. Let G be a finite group and $A \subseteq G$ a subset. Then there exists $g \in G$ such that
$$|A \cap Ag| \leq \frac{|A|^2}{|G|}.$$