Algebra Problem Solving Seminar

Miklós Abért and Péter Frenkel

2013/2014 second semester, Sheet 2

1. Can the polynomial

$$\sum_{i=1}^{n} x_i^2 \sum_{j=1}^{n} y_j^2 - \left(\sum_{k=1}^{n} x_k y_k \right)^2$$

   be written as a sum of squares of polynomials with real coefficients?

2. For what n does it hold that all the coefficients of the cyclotomic polynomial $\Phi_n(x)$ are 0 or 1?

3. Let K be a field of characteristic different from 2 and let $A, B \in M_n(K)$. Then the matrices

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix} \text{ and } \begin{pmatrix} A+B & 0 \\ 0 & A-B \end{pmatrix}$$

   are conjugate in $M_{2n}(K)$.

4. A basic step on a pair (a, b) of integers is to add an integer multiple of one of the entries to the other entry. Can you reach $(0, x)$ from all pairs of integers in 1000000 basic steps?

5. A finite group can be generated by a conjugacy class if and only if G/G' is cyclic.

6. What is the maximal order of an Abelian subgroup of $\mathrm{Sym}(n)$?

7. Let p be a prime. Then every subgroup of $\mathrm{Sym}(p)$ generated by p-cycles is simple.

8. Let a, b be nontrivial commuting elements of the free group F. Then there exists $c \in F$ and integers n, m, such that $c^n = a$ és $c^m = b$.

9. Let Γ be a finitely generated matrix group over the complex numbers. Then the intersection of finite index subgroups in Γ equals 1.