Algebra Problem Solving Seminar

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1. A basic step on a triple (a, b, c) of integers is to add an integer multiple of one of the entries to another entry. Can you reach $(0, 0, x)$ from all triples of integers in 5 basic steps?

2. Let $\omega : M_3(\mathbb{C}) \times M_3(\mathbb{C}) \to \mathbb{C}$ be an antisymmetric bilinear form such that $\omega(\mathbf{1}, g) = 0$ for all $g \in M_3(\mathbb{C})$. Does it follow that for all natural numbers k and l,
$$\frac{\omega(g^k, g^l)}{\omega(g, g^2)} = \frac{\omega(h^k, h^l)}{\omega(h, h^2)}$$
whenever g and h are similar matrices such that both denominators are nonzero?

3. (a) If $f \in \mathbb{R}[x]$ is nonnegative on $\mathbb{R}$, then it is a sum of the squares of two polynomials.

   (b) By definition, a trigonometric polynomial is a linear combination of the functions $\cos nx$, $\sin nx$ $(n = 0, 1, \dots)$.

   Prove that if a trigonometric polynomial is nonnegative on $\mathbb{R}$, then it is a sum of the squares of two trigonometric polynomials.

4. In the group $\mathrm{SL}_2(\mathbb{Z})$, the elements
$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$
generate a free subgroup.

5. (a) No element of $\mathbb{Z}$ is irreducible in the ring
$$\{a + bi + cj + dk | a, b, c, d \in \mathbb{Z} \text{ or } a, b, c, d \in \mathbb{Z} + 1/2\} < \mathbb{H}.$$

   (b) Deduce that every natural number is a sum of four squares.

6. Let η be a group endomorphism such that $\operatorname{im} \eta^m = \operatorname{im} \eta^{m+1}$ and $\ker \eta^k = \ker \eta^{k+1}$ for some m and k. Prove that the minimal such m and k is the same number.

7. For a finite extension L of a finite field K, both trace and norm are surjective $L \to K$ maps. By definition, the trace of x is $\sum \sigma(x)$ and the norm of x is $\prod \sigma(x)$, where σ runs over the Galois group.

8. $\mathrm{PSL}_2(4) \simeq A_5$.