Algebra Problem Solving Seminar

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1. Let G be a finite group of order n. A k-tuple $(a_1, \ldots, a_k) \in G^k$ is generating, if $a_1, \ldots, a_k$ generate G. A basic step on a k-tuple $(a_1, \ldots, a_k) \in G^k$ is to multiply an entry with another entry or its inverse from the right. Show that if $k > 2\log_2 n$ then any two generating k-tuples can be reached from one another in basic steps.

2. Let G be an infinite connected vertex transitive graph with finite degree. Show that G admits an infinite geodesic (infinite in both directions).

3. Let p be an odd prime. Show that the Cayley graph of $\mathrm{SL}_2(\mathbb{F}_p)$ with respect to the generating set
$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$
has girth $O(\log p)$.

4. Let T denote the 3-regular tree. Show that every automorphism of T either fixes a vertex, or an edge, or an infinite geodesic (as a set).

5. Show that for every $c > 0$ there are finitely many finite groups with c conjugacy classes.

6. Let S be a nonabelian finite simple group. Let M be an $k \times n$ matrix with entries in S such that all the columns generate S. Furthermore, assume that for any two distinct columns of M, there exists no automorphism of S that brings one column to the other. Prove that the rows of M generate the full S^n.

7. Let G be a d-regular graph. Show that the property that the infinite random walk on G almost surely returns to its starting vertex is independent of the starting point.

8. Let $\mathrm{FAlt}(N)$ denote the group of even (finitely supported) permutations on the natural numbers. Show that $\mathrm{FAlt}(N)$ is a simple group.

9. Prove that two independent random elements in $\mathrm{Sym}(n)$ generate a transitive permutation group with probability tending to 1 (as n tends to infinity).