Algebra Problem Solving Seminar

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1. Can every finite perfect group be generated by 2 elements? Recall that a group is perfect if it equals its derived subgroup.

2. Let A be an Abelian subgroup of the finitary symmetric group $\mathrm{FSym}(\mathbb{N})$. Show that every orbit of A is finite. How about nilpotent subgroups?

3. Let G be a d-regular connected graph and let v be a vertex of G. Let B_n denote the ball of radius n centered at v and let c_n denote the number of infinite connected components of the subgraph spanned on $V(G)\setminus B_n$. Let the number of ends be the limit of c_n. Show that when G is infinite and vertex transitive, the number of ends of G is $1, 2$ or infinity.

4. Let C denote the unit circle with Lebesque measure λ and let α be an irrational rotation of C. Let $A \subseteq C$ be a measurable subset of C such that $a(A) = A$. Show that $\lambda(A) = 0$ or $\lambda(C\setminus A) = 0$.

5. Let T denote the 3-regular tree. Let p_1,p_2 be disjoint bi-infinite geodesics in T and let g_i be an automorphism of T that acts on p_i by a nontrivial translation $(i = 1, 2)$. Prove that g_1 and g_2 generate a free group.

6. Let G be a d-regular graph and let $v \in V(G)$. Let p be the probability that the infinite random walk on G starting at v visits v again and let E be the expected number of returns to v. Show that
$$E = \frac{1}{1-p}.$$

7. Let G_n be a sequence of finite simple groups and let $\phi_n : G_n \to G_{n+1}$ be injective homomorphisms. Show that the limit $\cup_n G_n$ along ϕ_n is a simple group.

8. Prove that two independent random elements in $\mathrm{Sym}(n)$ generate a primitive permutation group with probability tending to 1 (as n tends to infinity). Recall that a permutation group is primitive if it admits no nontrivial invariant partition.