Algebra Problem Solving Seminar

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1. Let $f \in \mathbb{R}[x_1, \dots, x_n]$.

   (a) We have $f \geq 0$ on $[0, \infty)^n$ if and only if $f(x_1^2, \dots, x_n^2) \geq 0$ on $\mathbb{R}^n$.

   (b) We can write f as a finite sum $\sum f_j g_j^2$ with $f_j, g_j \in \mathbb{R}[x_1, \dots, x_n]$, all coefficients of f_j being nonnegative, if and only if $f(x_1^2, \dots, x_n^2)$ is a sum of squares in $\mathbb{R}[x_1, \dots, x_n]$.

2. Let S be an alphabet such that for every $s \in S$ the formal inverse $s^{-1} \in S$. Two finite words in S are equivalent if one can get one from the other by erasing a subword of the form ss^{-1} or $s^{-1}s$. This extends to an equivalence relation on finite words in S. Prove that every class contains exactly one word with no subwords of the form ss^{-1} or $s^{-1}s$. (In other words, prove that the free group F_S is well defined.)

3. A basic step on a triple (a, b, c) of integers is to add an integer multiple of one of the entries to another entry. Can you reach $(0, 0, x)$ from all triples of integers in 1000000 basic steps?

4. Let $g \in M_2(\mathbb{C})$ and
$$g^l = \begin{pmatrix} a_l & b_l \\ c_l & d_l \end{pmatrix}.$$
   Prove that b_l/b_1 does not change if we replace g by a matrix similar to it.

5. Prove that every finite group can be obtained as the automorphism group of a finite graph.

6. Let K be a field and $A \in M_n(K)$. Then A and $A^\top$ are similar over K.

7. Prove that every finite group of order n can be generated by at most $\log_2 n$ elements. When do we have equality?

8. Does there exist a nontrivial Abelian group $(A, +)$ such that for all rings $(A, +, \cdot)$, the multiplication $\cdot$ is identically zero?

9. The group of units of the ring
$$\{a + bi + cj + dk | a, b, c, d \in \mathbb{Z} \text{ or } a, b, c, d \in \mathbb{Z} + 1/2\} < \mathbb{H}$$
   is isomorphic to $\mathrm{SL}_2(3)$.